Backstepping Control Design for the Adaptive Stabilization and Synchronization of the Pandey Jerk Chaotic System with Unknown Parameters

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Abstract: This paper derives new results for the global chaos control and synchronization of the Pandey jerk chaotic system (2012) with unknown parameters via adaptive backstepping control method. First, this paper describes the dynamic equations, phase portraits and qualitative properties of the Pandey jerk chaotic system. The Pandey jerk chaotic system has two equilibrium points along the \( x_1 \) axis, which are both saddle-foci and unstable. The Lyapunov exponents of the Pandey jerk chaotic system are obtained as \( L_1 = 0.1148 \), \( L_2 = 0 \) and \( L_3 = -0.5363 \). Thus, the maximal Lyapunov exponent (MLE) of the Pandey jerk chaotic system is derived as \( L_1 = 0.1148 \). Since the sum of the Lyapunov exponents of the Pandey jerk chaotic system is dissipative, this jerk system is dissipative. Also, the Kaplan-Yorke dimension of the Pandey jerk chaotic system is derived as \( D_{\text{KY}} = 2.2141 \). Next, an adaptive controller is designed via backstepping control method to globally stabilize the Pandey jerk chaotic system with unknown parameters. Moreover, an adaptive controller is also designed via backstepping control method to achieve global and exponential synchronization of the identical Pandey jerk chaotic systems with unknown parameters. The main adaptive backstepping control results for stabilization and synchronization of the Pandey jerk chaotic system are established using Lyapunov stability theory.

Keywords: Chaos, chaotic systems, jerk systems, Pandey system, chaos control, chaos synchronization, backstepping control, stability theory.

1. INTRODUCTION

Chaos theory describes the qualitative study of unstable aperiodic behaviour in deterministic nonlinear dynamical systems. A dynamical system is called \textit{chaotic} if it satisfies the three properties: boundedness, infinite recurrence and sensitive dependence on initial conditions [1]. Chaos theory has applications in several areas in Science and Engineering.

A significant development in chaos theory occurred when Lorenz discovered a 3-D chaotic system of a weather model [2]. Subsequently, Rössler found a 3-D chaotic system [3], which is algebraically simpler than the Lorenz system. Indeed, Lorenz’s system is a seven-term chaotic system with two quadratic nonlinearities, while Rössler’s system is a seven-term chaotic system with just one quadratic nonlinearity.

Some well-known paradigms of 3-D chaotic systems are Arneodo system [4], Sprott systems [5], Chen system [6], Lü-Chen system [7], Liu system [8], Cai system [9], Tigan system [10], etc.

In the last two decades, many new chaotic systems have been also discovered like Li system [11], Sundarapandian systems [12-13], Vaidyanathan systems [14-33], Pehlivan systems [34-35], Pham systems [36-37], Jafari system [38], etc.

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Hyperchaotic systems are the chaotic systems with more than one positive Lyapunov exponent. They have important applications in control and communication engineering. Some recently discovered 4-D hyperchaotic systems are hyperchaotic Vaidyanathan systems [39-40], hyperchaotic Vaidyanathan-Azar system [41], etc. A 5-D hyperchaotic system with three positive Lyapunov exponents was also recently found [42].

Chaos theory has several applications in a variety of fields such as oscillators [43-44], chemical reactors [45-58], biology [59-80], ecology [81-82], neural networks [83-84], robotics [85-86], memristors [87-89], fuzzy systems [90-91], etc.

The problem of control of a chaotic system is to find a state feedback control law to stabilize a chaotic system around its unstable equilibrium [92-93]. Some popular methods for chaos control are active control [94-98], adaptive control [99-100], sliding mode control [101-103], etc.

Chaos synchronization problem can be stated as follows. If a particular chaotic system is called the master or drive system and another chaotic system is called the slave or response system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically. The synchronization of chaotic systems has applications in secure communications [104-107], cryptosystems [108-109], encryption [110-111], etc.

The chaos synchronization problem has been paid great attention in the literature and a variety of impressive approaches have been proposed. Since the pioneering work by Pecora and Carroll [112-113] for the chaos synchronization problem, many different methods have been proposed in the control literature such as active control method [114-132], adaptive control method [133-149], sampled-data feedback control method [150-151], time-delay feedback approach [152], backstepping method [153-164], sliding mode control method [165-173], etc.

In the recent decades, there is some good interest in finding novel chaotic systems, which can be expressed by an explicit third order differential equation describing the time evolution of the single scalar variable $x$ given by

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) \tag{1}$$

The differential equation (1) is called “jerk system” because the third order time derivative in mechanical systems is called jerk. Thus, in order to study different aspects of chaos, the ODE (1) can be considered instead of a 3-D system.

In this paper, we describe the properties of the Pandey 3-D jerk chaotic system ([174], 2012) with a single quadratic nonlinearity. The Lyapunov exponents of the Pandey jerk chaotic system are obtained as $L_1 = 0.1148$, $L_2 = 0$ and $L_3 = -0.5363$. The Kaplan-Yorke dimension of the Pandey system is derived as $D_{KY} = 2.2141$.

Next, using backstepping control method, we derive an adaptive control law that stabilizes the Pandey jerk chaotic system, when the system parameters are unknown. Using backstepping control method, we also derive an adaptive control law that achieves global chaos synchronization of the identical novel jerk chaotic systems with unknown parameters. Also, this paper derives an adaptive control law that stabilizes the Pandey jerk chaotic system with unknown system parameters. This paper also derives an adaptive control law that achieves global chaos synchronization of identical Pandey chaotic systems with unknown parameters.

This paper is organized as follows. In Section 2, we describe the Pandey jerk chaotic system with a single quadratic nonlinearity. In Section 3, we describe the qualitative properties of the Pandey jerk chaotic system. In Section 4, we detail the adaptive backstepping control design for the global chaos
stabilization of the Pandey jerk chaotic system with unknown parameters. In Section 5, we detail the adaptive backstepping control design for the global and exponential synchronization of the identical Pandey jerk chaotic systems.

2. PANDEY JERK CHAOTIC SYSTEM

In this section, we describe the Pandey jerk chaotic system [174] with a single quadratic nonlinearity, which is modeled by the 3-D dynamics

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -ax_1 - bx_2 - cx_3 - x_1^2
\end{align*}
\]  

(2)

where \(x_1, x_2, x_3\) are state variables and \(a, b, c\) are constant, positive, parameters of the system.

The Pandey jerk system (2) exhibits a strange chaotic attractor for the values

\[a = 1, \quad b = 1.1, \quad c = 0.42\]

(3)

For numerical simulations, we take the initial conditions of the state as

\[x_1(0) = 0.1, \quad x_2(0) = 0.1, \quad x_3(0) = 0.1\]

(4)

The Lyapunov exponents of the jerk chaotic system (2) for the parameter values (3) and the initial conditions (4) are numerically calculated as

\[L_1 = 0.1148, \quad L_2 = 0, \quad L_3 = -0.5363\]

(5)

Figure 1 shows the 3-D phase portrait of the Pandey jerk chaotic system (2). Figures 2-4 show the 2-D projection of the Pandey jerk system (2) on the \((x_1, x_2), (x_2, x_3),\) and \((x_1, x_3)\) planes, respectively.

3. PROPERTIES OF THE PANDEY JERK CHAOTIC SYSTEM

In this section, we shall discuss the qualitative properties of the Pandey jerk chaotic system (2) described in Section 2. We suppose that the parameter values of the Pandey jerk system (2) are as in the chaotic case (3), i.e. \(a = 1, \quad b = 1.1\) and \(c = 0.42.\)
3.1. Dissipativity of the Flow

In vector notation, we may express the system (2) as

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{bmatrix}$$

(6)

where

$$\begin{align*}
f_1(x_1, x_2, x_3) &= x_2 \\
f_2(x_1, x_2, x_3) &= x_3 \\
f_3(x_1, x_2, x_3) &= -ax_1 - bx_2 - cx_3 - x_1^2
\end{align*}$$

(7)

Let $\Omega$ be any region in $\mathbb{R}^3$ with a smooth boundary and also $\Omega(t) = \Phi_t(\Omega)$, where $\Phi_t$ is the flow of the vector field $f$. Furthermore, let $V(t)$ denote the volume of $\Omega(t)$.

By Liouville’s theorem, we have

$$\dot{V} = \int_{\Omega(t)} (\nabla \cdot f) \, dx_1 \, dx_2 \, dx_3$$

(8)

The divergence of the Pandey jerk chaotic system (2) is easily found as

$$\nabla \cdot f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = -c$$

(9)

Substituting (9) into (8), we obtain the first order ODE

$$\dot{V} = \int_{\Omega(t)} (-c) \, dx_1 \, dx_2 \, dx_3 = -c V$$

(10)

Integrating (10), we obtain the unique solution as

$$V(t) = \exp(-ct) \, V(0) \text{ for all } t \geq 0$$

(11)
Since \( c > 0 \), we conclude from Eq. (11) that \( V(t) \to 0 \) exponentially as \( t \to \infty \).

This shows that the Pandey jerk chaotic system (2) is dissipative. Hence, the system limit sets are ultimately confined into a specific limit set of zero volume, and the asymptotic motion of the Pandey jerk chaotic system (2) settles onto a strange attractor of the system.

### 3.2. Equilibrium Points

We take the values of the parameters as in the chaotic case (3), i.e. \( a = 1 \), \( b = 1.1 \) and \( c = 0.42 \).

The equilibrium points of the Pandey jerk system (2) are obtained by solving the system of equations

\[
\begin{align*}
x_2 &= 0 \\
x_3 &= 0 \\
-ax_1 - bx_2 - cx_3 - x_1^2 &= 0
\end{align*}
\]  

Solving the system (12), we obtain two equilibrium points given by

\[
E_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}
\]  

(13)

The Jacobian of the Pandey jerk chaotic system (2) at any point \( x \in \mathbb{R}^3 \) is given by

\[
J(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 - 2x_1 & -1.1 & -0.42 \end{bmatrix}
\]  

(14)

The Jacobian of the Pandey jerk chaotic system (2) at \( E_0 \) is obtained as

\[
J_0 = J(E_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1.1 & -0.42 \end{bmatrix}
\]  

(15)

The eigenvalues of \( J_0 \) are numerically obtained as

\[
\lambda_1 = -0.7451, \quad \lambda_{2,3} = 0.1625 \pm 1.1471i
\]  

(16)

This shows that \( E_0 \) is a saddle-focus, which is unstable.

Next, the Jacobian of the jerk chaotic system (2) at \( E_1 \) is obtained as

\[
J_1 = J(E_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1.1 & -0.42 \end{bmatrix}
\]  

(17)

The eigenvalues of the matrix \( J_1 \) are numerically obtained as

\[
\lambda_1 = 0.5898, \quad \lambda_{2,3} = -0.5049 \pm 1.2003i
\]  

(18)

This shows that \( E_1 \) is a saddle-focus, which is unstable.
3.4 Lyapunov Exponents and Kaplan-yorke Dimension

We take the parameter values of the Pandey jerk system (2) as in the chaotic case (3), i.e.

\[ a = 1, \ b = 1.1, \ c = 0.42 \]  \hspace{1cm} (19)

We choose the initial values of the Pandey jerk system (2) as

\[ x_1(0) = 0.1, \ x_2(0) = 0.1, \ x_3(0) = 0.1 \]  \hspace{1cm} (20)

Then we obtain the Lyapunov exponents of the Pandey jerk system (2) as

\[ L_1 = 0.1148, \ L_2 = 0, \ L_3 = -0.5363 \]  \hspace{1cm} (21)

Figure 5 shows the Lyapunov exponents of the system (2) as determined by MATLAB.

We note that the sum of the Lyapunov exponents of the Pandey jerk system (2) is negative, which shows that the Pandey jerk system (2) is dissipative.

Also, the Maximal Lyapunov Exponent (MLE) of the jerk chaotic system (2) is \( L_1 = 0.1148 \).

The Kaplan-Yorke dimension of the Pandey jerk chaotic system (2) is derived as

\[ D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.2141 \]  \hspace{1cm} (22)

4. ADAPTIVE BACKSTEPPING CONTROL DESIGN FOR THE STABILIZATION OF THE PANDEY JERK CHAOTIC SYSTEM

In this section, we consider the novel jerk system with a single control given by

![Figure 5: Lyapunov exponents of the Pandey jerk chaotic system](image)
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -ax_1 - bx_2 - cx_3 - x_1^2 + u
\end{align*}
\] (23)

In (23), \(x_1, x_2, x_3\) are the states, \(a, b, c\) are unknown constant parameters, and \(u\) is a backstepping control law to be determined using estimates \(\hat{a}(t), \hat{b}(t), \hat{c}(t)\) of the unknown parameters \(a, b, c\) respectively.

The parameter estimation errors are defined as follows:
\[
\begin{align*}
e_a(t) &= a - \hat{a}(t) \\
e_b(t) &= b - \hat{b}(t) \\
e_c(t) &= c - \hat{c}(t)
\end{align*}
\] (24)

Differentiating (24) with respect to \(t\), we obtain
\[
\begin{align*}
\dot{e}_a(t) &= -\hat{a}(t) \\
\dot{e}_b(t) &= -\hat{b}(t) \\
\dot{e}_c(t) &= -\hat{c}(t)
\end{align*}
\] (25)

Next, we shall state and prove the main result of this section.

**Theorem 1.** The Pandey jerk chaotic system (23) with unknown parameters is globally and exponentially stabilized by the adaptive feedback control law
\[
\begin{align*}
u &= -\left[3 - \hat{a}(t)\right]x_1 - \left[5 - \hat{b}(t)\right]x_2 - \left[3 - \hat{c}(t)\right]x_3 + x_1^2 - kz_3
\end{align*}
\] (26)

where \(k > 0\) is a gain constant, with
\[
z_3 = 2x_1 + 2x_2 + x_3
\] (27)

and the parameter update law is given by
\[
\begin{align*}
\dot{\hat{a}} &= -x_1z_3 \\
\dot{\hat{b}} &= -x_2z_3 \\
\dot{\hat{c}} &= -x_3z_3
\end{align*}
\] (28)

**Proof.** We prove this result via backstepping control method and Lyapunov stability theory [175].

First, we define a quadratic Lyapunov function
\[
V_1(z_1) = \frac{1}{2}z_1^2
\] (29)

where
\[
z_1 = x_1
\] (30)
Differentiating $V_1$ along the dynamics (23), we obtain

$$\dot{V}_1 = x_i x_2 = -z_i^2 + z_i (x_i + x_2)$$

(31)

Now we define

$$z_2 = x_i + x_2$$

(32)

Using (32), we can simplify (31) as

$$\dot{V}_1 = -z_i^2 + z_i z_2$$

(33)

Next, we define a quadratic Lyapunov function

$$V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_i^2 = \frac{1}{2} \left( z_i^2 + z_2^2 \right)$$

(34)

Differentiating $V_2$ along the dynamics (23), we obtain

$$\dot{V}_2 = -z_1^2 - z_2^2 + z_i (2x_i + 2x_2 + x_3)$$

(35)

Now, we define

$$z_3 = 2x_i + 2x_2 + x_3$$

(36)

Using (36), we can simplify (35) as

$$\dot{V}_2 = -z_i^2 - z_2^2 + z_2 z_3$$

(37)

Finally, we define a quadratic Lyapunov function

$$V(z, e_a, e_b, e_c) = V_2(z_1, z_2) + \frac{1}{2} z_i^2 + \frac{1}{2} (e_a^2 + e_b^2 + e_c^2)$$

(38)

From (38), it is clear that $V$ is a positive definite function on $\mathbb{R}^6$.

Differentiating $V$ along the dynamics (23) and (28), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_i S - e_a \dot{a} - e_b \dot{b} - e_c \dot{c}$$

(39)

In (39), $S$ is given by

$$S = z_1 + z_2 + z_3 = z_1 + z_2 + 2\dot{x}_1 + 2\dot{x}_2 + \dot{x}_3$$

(40)

Simplifying the equation (40), we obtain

$$S = (3 - a)x_1 + (5 - b)x_2 + (3 - c)x_3 - x_1^2 + u$$

(41)

Substituting the control law (26) into (41), we get

$$S = -[a - \dot{a}(t)]x_1 - [b - \dot{b}(t)]x_2 - [c - \dot{c}(t)]x_3 - k z_3$$

(42)

Using the definitions in (24), we can simplify the equation (42) as

$$S = -e_a x_1 - e_b x_2 - e_c x_3 - k z_3$$

(43)

Substituting (43) into (39), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - (1 + k) z_3^2 + e_a \left[ -x_i z_3 - \dot{a} \right] + e_b \left[ -x_2 z_3 - \dot{b} \right] + e_c \left[ -x_3 z_3 - \dot{c} \right]$$

(44)
Backstepping Control Design for the Adaptive Stabilization and Synchronization...

Substituting the parameter update law (28) into (44), we obtain

$$\dot{V} = -z_1^2 - z_2^2 - (1 + k)z_3^2$$

(45)

Thus, it is clear that $\dot{V}$ is a negative semi-definite function on $\mathbb{R}^6$.

From (45), it is clear that the vector $z(t) = (z_1(t), z_2(t), z_3(t))$ and the parameter estimation error $(e_a(t), e_b(t), e_c(t))$ are globally bounded, i.e.

$$\begin{bmatrix} z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) & e_c(t) \end{bmatrix}^T \in \mathbf{L}_c$$

(46)

Also, it follows from (45) that

$$\dot{V} \leq -z_1^2 - z_2^2 - z_3^2 = -\|z\|^2$$

(47)

or

$$\|z(t)\|^2 \leq -\dot{V}(t)$$

(48)

Integrating the inequality (48) from 0 to $t$, we get

$$\int_0^t \|z(\tau)\|^2 d\tau \leq V(0) - V(t)$$

(49)

From (49), it follows that $z(t) \in L_2^\infty$.

From (23), it can be deduced that $z(t) \in L_c^\infty$.

Thus, using Barbalat’s lemma [175], we can conclude that $z(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $z(0) \in \mathbb{R}^3$.

Hence, it is immediate that $x(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $x(0) \in \mathbb{R}^3$.

This completes the proof. ■

For numerical simulations, the classical fourth-order Runge-Kutta method with step size $h = 10^{-8}$ is used to solve the system of differential equations (23) and (28), when the adaptive controller (26) is implemented.

The parameter values of the Pandey jerk chaotic system (23) are taken as in the chaotic case, i.e.

$$a = 1, \quad b = 1.1, \quad c = 0.42$$

(50)

The positive gain constant $k$ is taken as $k = 10$.

The initial conditions of the Pandey jerk system (23) are taken as

$$x_1(0) = 12.8, \quad x_2(0) = 17.3, \quad x_3(0) = -5.4$$

(51)

The initial conditions of the parameter estimates are taken as

$$\hat{a}(0) = 3.4, \quad \hat{b}(0) = 5.7, \quad \hat{c}(0) = 6.2$$

(52)

Figure 6 shows the time-history of the controlled states $x_1(t), x_2(t), x_3(t)$.

5. ADAPTIVE BACKSTEPPING CONTROL DESIGN FOR THE GLOBAL CHAOS SYNCHRONIZATION OF THE IDENTICAL PANDEY JERK CHAOTIC SYSTEMS

In this section, we use backstepping control method to derive an adaptive feedback control law for globally synchronizing identical Pandey jerk chaotic systems with unknown parameters.
As the master system, we consider the novel jerk system given by
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -ax_1 - bx_2 - cx_3 - x_1^2
\end{align*}
\] (53)

In (53), \(x_1, x_2, x_3\) are the states and \(a, b, c\) are unknown system parameters.

As the slave system, we consider the controlled novel jerk system given by
\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\dot{y}_3 &= -ay_1 - by_2 - cy_3 - y_1^2 + u
\end{align*}
\] (54)

In (54), \(y_1, y_2, y_3\) are the states and \(u\) is the adaptive control to be determined using estimates of the unknown system parameters.

The complete synchronization error between the systems (53) and (54) is defined by
\[
\begin{align*}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 - x_2 \\
e_3 &= y_3 - x_3
\end{align*}
\] (55)

Then the synchronization error dynamics is obtained as
\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= e_3 \\
\dot{e}_3 &= -ae_1 - be_2 - ce_3 - y_1^2 + x_1^2 + u
\end{align*}
\] (56)
The parameter estimation errors are defined as follows:

\[
\begin{align*}
  e_a(t) &= a - \hat{a}(t) \\
  e_b(t) &= b - \hat{b}(t) \\
  e_c(t) &= c - \hat{c}(t)
\end{align*}
\]  

(57)

Differentiating (57) with respect to \( t \), we obtain

\[
\begin{align*}
  \dot{e}_a(t) &= -\hat{a}(t) \\
  \dot{e}_b(t) &= -\dot{\hat{b}}(t) \\
  \dot{e}_c(t) &= -\dot{\hat{c}}(t)
\end{align*}
\]  

(58)

Next, we shall state and prove the main result of this section.

**Theorem 2.** The Pandey jerk chaotic systems (53) and (54) with unknown parameters is globally and exponentially synchronized by the adaptive feedback control law

\[
u = -[3 - \hat{a}(t)]e_1 - [5 - \dot{\hat{b}}(t)]e_2 - [3 - \dot{\hat{c}}(t)]e_3 + y_i^2 - x_i^2 - kz_3
\]  

(59)

where \( k > 0 \) is a gain constant, with

\[
z_3 = 2e_1 + 2e_2 + e_3
\]  

(60)

and the parameter update law is given by

\[
\begin{align*}
  \dot{\hat{a}} &= -e_1z_3 \\
  \dot{\hat{b}} &= -e_2z_3 \\
  \dot{\hat{c}} &= -e_3z_3
\end{align*}
\]  

(61)

**Proof.** We prove this result via backstepping control method and Lyapunov stability theory [175].

First, we define a quadratic Lyapunov function

\[
V_1(z_1) = \frac{1}{2} z_1^2
\]  

(62)

where

\[
z_1 = e_1
\]  

(63)

Differentiating \( V_1 \) along the dynamics (56), we obtain

\[
\dot{V}_1 = e_1e_2 = -z_1^2 + z_1(e_1 + e_2)
\]  

(64)

Now we define

\[
z_2 = e_1 + e_2
\]  

(65)

Using (65), we can simplify (64) as
Next, we define a quadratic Lyapunov function
\[ V_2(z_1, z_2) = V_1(z_1) + \frac{1}{2} z_2^2 = \frac{1}{2} \left( z_1^2 + z_2^2 \right) \]  
(67)
Differentiating \( V_2 \) along the dynamics (56), we obtain
\[ \dot{V}_2 = -z_1^2 - z_2^2 + z_2 (2e_1 + 2e_2 + e_3) \]  
(68)
Now, we define
\[ z_3 = 2x_1 + 2x_2 + x_3 \]  
(69)
Using (69), we can simplify (68) as
\[ \dot{V}_2 = -z_1^2 - z_2^2 + z_2 z_3 \]  
(70)
Finally, we define a quadratic Lyapunov function
\[ V(z, e_a, e_b, e_c) = V_2(z_1, z_2) + \frac{1}{2} z_3^2 + \frac{1}{2} (e_a^2 + e_b^2 + e_c^2) \]  
(71)
From (71), it is clear that \( V \) is a positive definite function on \( \mathbb{R}^6 \).
Differentiating \( V \) along the dynamics (56) and (58), we obtain
\[ \dot{V} = -z_1^2 - z_2^2 - z_3^2 + z_3 S - e_a \dot{a} - e_b \dot{b} - e_c \dot{c} \]  
(72)
where
\[ S = z_3 + z_2 + \dot{z}_2 = z_3 + z_2 + 2\dot{e}_1 + 2\dot{e}_2 + \dot{e}_3 \]  
(73)
Simplifying the equation (73), we obtain
\[ S = (3 - a)e_1 + (5 - b)e_2 + (3 - c)e_3 - y_1^2 + x_1^2 + u \]  
(74)
Substituting the control law (59) into (74), we get
\[ S = -[a - \dot{a}(t)]e_1 - [b - \dot{b}(t)]e_2 - [c - \dot{c}(t)]e_3 - k z_3 \]  
(75)
Using the definitions in (57), we can simplify the equation (75) as
\[ S = -e_a e_1 - e_b e_2 - e_c e_3 - k z_3 \]  
(76)
Substituting (76) into (72), we obtain
\[ \dot{V} = -z_1^2 - z_2^2 - (1 + k) z_3^2 + e_a \left[ -e_1 z_3 - \dot{a} \right] + e_b \left[ -e_2 z_3 - \dot{b} \right] + e_c \left[ -e_3 z_3 - \dot{c} \right] \]  
(77)
Substituting the parameter update law (61) into (77), we obtain
\[ \dot{V} = -z_1^2 - z_2^2 - (1 + k) z_3^2 \]  
(78)
Thus, it is clear that \( \dot{V} \) is a negative semi-definite function on \( \mathbb{R}^6 \).
From (78), it is clear that the vector \( z(t) = (z_1(t), z_2(t), z_3(t)) \) and the parameter estimation error \( (e_a(t), e_b(t), e_c(t)) \), are globally bounded, i.e.
Backstepping Control Design for the Adaptive Stabilization and Synchronization...

\[
\begin{bmatrix}
    z_1(t) & z_2(t) & z_3(t) & e_a(t) & e_b(t) & e_c(t)
\end{bmatrix}^T \in L_\infty
\]

Also, it follows from (78) that

\[\dot{V} \leq -z_1^2 - z_2^2 - z_3^2 = -\|z\|^2\]

or

\[\|z(t)\|^2 \leq -\dot{V}(t)\]

Integrating the inequality (81) from 0 to \(t\), we get

\[\int_0^t \|z(\tau)\|^2 d\tau \leq V(0) - V(t)\]

From (82), it follows that \(z(t) \in L_2\).

From (56), it can be deduced that \(z(t) \in L_\infty\).

Thus, using Barbalat’s lemma [175], we can conclude that \(z(t) \to 0\) exponentially as \(t \to \infty\) for all initial conditions \(z(0) \in R^3\).

Hence, it is immediate that \(e(t) \to 0\) exponentially as \(t \to \infty\) for all initial conditions \(e(0) \in R^3\).

This completes the proof. \(\blacksquare\)

For numerical simulations, the classical fourth-order Runge-Kutta method with step-size \(h = 10^{-8}\) is used to solve the systems (53), (54) and (61), when the adaptive control law (59) is applied.

We take the parameter values of the Pandey jerk systems (53) and (54) as in the chaotic case, i.e. \(a = 1, b = 1.1\) and \(c = 0.42\). We take the positive gain constant as \(k = 10\).

As initial conditions of the master system (53), we take

\[x_1(0) = 0.1, \quad x_2(0) = 0.1, \quad x_3(0) = 0.2\]

As initial conditions of the slave system (54), we take

\[y_1(0) = 0.5, \quad y_2(0) = 0.3, \quad y_3(0) = -0.5\]

![Figure 7: Synchronization of the states \(x_1\) and \(y_1\)](image)

![Figure 8: Synchronization of the states \(x_2\) and \(y_2\)](image)
As initial conditions of the parameter estimates, we take
\[ \hat{a}(0) = 3.1, \quad \hat{b}(0) = 2.7, \quad \hat{c}(0) = 5.4 \]  
(85)

Figures 7-9 depict the synchronization of the Pandey jerk chaotic systems (53) and (54).

Figure 10 depicts the time-history of the complete synchronization errors \( e_1, e_2, e_3 \).

6. CONCLUSIONS

In this paper, we derived new results for the global chaos control and synchronization of the Pandey jerk chaotic system (2012) with unknown parameters via adaptive backstepping control method. The main adaptive backstepping control results for stabilization and synchronization of the Pandey jerk chaotic system were established using Lyapunov stability theory. MATLAB plots have been shown to illustrate the qualitative properties and adaptive control results for the Pandey jerk chaotic system.

References


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