Hybrid Synchronization of Identical Chaotic Systems via Novel Sliding Control Method with Application to Sampath Four-Scroll Chaotic System

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Abstract: First, this paper proposes a general procedure for the hybrid synchronization of identical chaotic systems using novel sliding mode control method. The general result derived using novel sliding mode control method is established via Lyapunov stability theory. As an application of the general result, the problem of hybrid synchronization of identical Sampath four-scroll chaotic systems (2015) is studied and a new sliding mode controller is derived. Numerical simulations with MATLAB have been shown to illustrate all the main results derived in this work.

Keywords: Chaos, chaotic systems, chaos synchronization, hybrid synchronization, sliding mode control.

1. INTRODUCTION

A chaotic system is commonly defined as a nonlinear dissipative dynamical system that is highly sensitive to even small perturbations in its initial conditions [1]. In other words, a chaotic system is a nonlinear dynamical system with at least one positive Lyapunov exponent. Some paradigms of chaotic systems can be listed as Arneodo system [4], Sprott systems [5], Chen system [6], Lü-Chen system [7], Liu system [8], Cai system [9], Tigan system [10], etc.

In the last two decades, many new chaotic systems have been also discovered like Li system [11], Sundarapandian systems [12-13], Vaidyanathan systems [14-33], Pehlivan systems [34-35], Pham systems [36-37], Jafari system [38], etc.

Hyperchaotic systems are the chaotic systems with more than one positive Lyapunov exponent. They have important applications in control and communication engineering. Some recently discovered 4-D hyperchaotic systems are hyperchaotic Vaidyanathan systems [39-40], hyperchaotic Vaidyanathan-Azar system [41], etc. A 5-D hyperchaotic system with three positive Lyapunov exponents was also recently found [42].

Chaos theory has several applications in a variety of fields such as oscillators [43-44], chemical reactors [45-58], biology [59-80], ecology [81-82], neural networks [83-84], robotics [85-86], memristors [87-89], fuzzy systems [90-91], etc.

The problem of control of a chaotic system is to find a state feedback control law to stabilize a chaotic system around its unstable equilibrium [92-93]. Some popular methods for chaos control are active control [94-98], adaptive control [99-100], sliding mode control [101-103], etc.

Chaos synchronization problem can be stated as follows. If a particular chaotic system is called the master or drive system and another chaotic system is called the slave or response system, then the idea of...
the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

The synchronization of chaotic systems has applications in secure communications [104-107], cryptosystems [108-109], encryption [110-111], etc.

The chaos synchronization problem has been paid great attention in the literature and a variety of impressive approaches have been proposed. Since the pioneering work by Pecora and Carroll [112-113] for the chaos synchronization problem, many different methods have been proposed in the control literature such as active control method [114-132], adaptive control method [133-149], sampled-data feedback control method [150-151], time-delay feedback approach [152], backstepping method [153-164], sliding mode control method [165-173], etc.

In this paper, new results have been derived for the hybrid chaos synchronization of identical chaotic systems using novel sliding control method. The sliding mode control method has advantages of low sensitivity to parameter variations in the plant and disturbances affecting the plant.

In Section 2, we describe the hybrid synchronization of identical chaotic systems. In Section 3, we derive a general result for the hybrid synchronization of identical chaotic systems using novel sliding control method. In Section 4, we describe the Sampath four-scroll chaotic system ([28], 2015) and its qualitative properties. The phase portraits of the Sampath four-scroll chaotic system are described using MATLAB. In Section 5, we describe the sliding mode controller design for the hybrid chaos synchronization of the identical Sampath four-scroll chaotic systems using novel sliding control method and its numerical simulations using MATLAB. Section 6 contains a summary of the main results derived in this paper.

2. HYBRID CHAOS SYNCHRONIZATION OF IDENTICAL CHAOTIC SYSTEMS

In this section, we provide a problem statement for the hybrid chaos synchronization of identical chaotic systems.

As the master system, we consider the chaotic system given by

\[ \dot{x} = Ax + \varphi(x) \]  

In Eq. (1), \( x \in \mathbb{R}^n \) denotes the state of the system, \( A \) denotes the matrix of system parameters and \( \varphi \) contains the nonlinear parts of the system.

As the slave or response system, we take the controlled chaotic system given by

\[ \dot{y} = Ay + \varphi(y) + u \]  

In Eq. (2), \( y \in \mathbb{R}^n \) denotes the state of the system and \( u \in \mathbb{R}^n \) is the control.

The hybrid chaos synchronization error between the systems (1) and (2) is defined as

\[ e_i = \begin{cases} y_i - x_i & \text{if } i \text{ is odd} \\ y_i + x_i & \text{if } i \text{ is even} \end{cases} \]  

A simple calculation yields the error dynamics as

\[ \dot{e} = Ae + \eta(x, y) + u \]  

Thus, the hybrid chaos synchronization problem for the chaotic systems (1) and (2) can be defined as follows: Find a controller \( u \) so as to render the hybrid chaos synchronization error \( e(t) \) to be globally asymptotically stable for all values of \( e(0) \in \mathbb{R}^n \), i.e.
\[ \lim_{t \to \infty} \|e(t)\| = 0 \text{ for all } e(0) \in \mathbb{R}^n. \]  

(5)

3. NOVEL SLIDING CONTROLLER DESIGN

First, we set the design by setting the control as

\[ u(t) = -\eta(x,y) + Bv(t) \]  

(6)

In Eq. (6), \(B \in \mathbb{R}^n\) is chosen such that \((A, B)\) is completely controllable.

By substituting (6) into (4), we get the closed-loop error dynamics

\[ \dot{e} = Ae + Bv \]  

(7)

The system (7) is a linear time-variant control system with single input \(v\).

Hence, the hybrid chaos synchronization of the identical chaotic systems (1) and (2) has been converted to an equivalent control problem of globally stabilizing the error system (7) by a suitable choice of the feedback control (6).

We start the sliding controller design by defining the sliding variable as

\[ s(e) = C e = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n, \]  

(8)

where \(C \in \mathbb{R}^{1 \times n}\) is a constant vector to be determined.

The sliding manifold \(S\) is defined as the hyperplane

\[ S = \{ e \in \mathbb{R}^n : s(e) = Ce = 0 \}. \]  

(9)

We shall assume that a sliding motion occurs on the hyperplane \(S\).

In sliding mode, the following equations must be satisfied:

\[ s = 0 \text{ and } \dot{s} = CAe + CBv = 0 \]  

(10)

We assume that

\[ CB \neq 0 \]  

(11)

The sliding motion is influenced by the equivalent control derived from (10) as

\[ v_{eq}(t) = -(CB)^{-1}CAe(t) \]  

(12)

By substituting (12) into (7), we obtain the equivalent error dynamics in the sliding phase as

\[ \dot{e} = Ae - (CB)^{-1}CAe = Ee, \]  

(13)

where

\[ E = [I - B(CB)^{-1}]A \]  

(14)

We note that \(E\) is independent of the control and has at most \((n - 1)\) nonzero eigenvalues, depending on the chosen switching surface, while the associated eigenvectors belong to \(\ker(C)\).

Since \((A, B)\) is controllable, we can use sliding control theory to choose \(B\) and \(C\) so that \(E\) has any desired \((n - 1)\) stable eigenvalues.

This shows that the dynamics in the sliding mode is globally asymptotically stable.

Finally, for the sliding controller design, we apply a novel sliding control law, viz.
In Eq. (15), sgn() denotes the sign function and the sliding mode control constants \( k > 0, q > 0 \) are found in such a way that the sliding condition is satisfied and that the sliding motion will occur.

By combining equations (10), (12) and (15), we finally obtain the sliding mode control (SMC) \( \nu(t) \) as

\[
\nu(t) = -(CB)^{-1}[C(kI + A)e + qs^2 \text{sgn}(s)]
\]

Next, we establish the main result of this section.

**Theorem 1.** The sliding mode controller law defined by (6) achieves global and asymptotic hybrid chaos synchronization of the identical chaotic systems (1) and (2) for all initial conditions \( x(0), y(0) \in \mathbb{R}^n \), where \( \nu \) is defined by the novel sliding control law (16), \( B \in \mathbb{R}^{n \times 1} \) is such that \( (A, B) \) is controllable, \( C \in \mathbb{R}^{1 \times n} \) is such that \( CB \neq 0 \) and that the matrix \( E \) defined by (14) has \( (n-1) \) stable eigenvalues.

**Proof.** Upon substitution of the control laws (6) and (16) into the error dynamics (4), we get the closed-loop error dynamics as

\[
\dot{e} = Ae - B(CB)^{-1} \left[ C(kI + A)e + qs^2 \text{sgn}(s) \right]
\]

We shall show that the error system (17) is globally asymptotically stable by considering the quadratic Lyapunov function

\[
V(e) = \frac{1}{2} s^2(e)
\]

The sliding mode motion is characterized by the equations

\[
s(e) = 0 \quad \text{and} \quad \dot{s}(e) = 0
\]

By the choice of \( E \), the dynamics in the sliding mode is globally asymptotically stable.

When \( s(e) \neq 0 \), \( V(e) > 0 \).

Also, when \( s(e) \neq 0 \), differentiating \( V \) along the error dynamics (17) or the equivalent dynamics (15), we get

\[
\dot{V}(e) = ss = -ks^2 - qs^3 \text{sgn}(s) < 0.
\]

Hence, by Lyapunov stability theory [174], the error dynamics (18) is globally asymptotically stable for all \( e(0) \in \mathbb{R}^n \). This completes the proof. \(

**4. SAMPATH FOUR-SCROLL CHAOTIC SYSTEM**

The Sampath four-scroll chaotic system ([28], 2015) is described by the 3-D dynamics

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) + bx_2x_3 \\
\dot{x}_2 &= -10x_2^3 - x_2 + 4x_1x_3 \\
\dot{x}_3 &= cx_3 - x_1x_2
\end{align*}
\]

where \( x_1, \ x_2, \ x_3 \) are state variables and \( a, b, c, p \) are constant, positive, parameters of the system.

The system (21) exhibits a *four-scroll chaotic attractor* for the values

\[
a = 3, \quad b = 14, \quad c = 3.9
\]
The Lyapunov exponents of the system (21) are numerically obtained with MATLAB as

\[ L_1 = 0.75535, \quad L_2 = 0, \quad L_3 = -22.43304 \]  \hspace{1cm} (23)

The Lyapunov dimension of the chaotic system (1) is determined as

\[ D_L = \frac{L_1 + L_2}{|L_3|} = 2.0336 \]  \hspace{1cm} (24)

For numerical simulations, the initial values of the Sampath system (21) are taken as

\[ x_1(0) = 0.3, \quad x_2(0) = 0.4, \quad x_3(0) = 0.3 \]  \hspace{1cm} (25)

Figure 1 shows the strange chaotic attractor of the Sampath four-scroll chaotic system (21).

Figures 2-4 show the 2-D view of the chaotic attractor of the system (21) in \((x_1, x_2)\), \((x_2, x_3)\), and \((x_1, x_3)\) planes respectively.

**Figure 1:** Four-scroll attractor of the Sampath four-scroll chaotic system

**Figure 2:** 2-D view of the Sampath four-scroll chaotic system in \((x_1, x_2)\) plane

**Figure 3:** 2-D view of the Sampath four-scroll chaotic system in \((x_2, x_3)\) plane

**Figure 4:** 2-D view of the Sampath four-scroll chaotic system in \((x_1, x_3)\) plane
5. HYBRID CHAOS SYNCHRONIZATION OF SAMPATH FOUR-SCROLL CHAOTIC SYSTEMS VIA NOVEL SLIDING CONTROLLER

In this section, we describe novel sliding controller design for the hybrid chaos synchronization of identical Sampath four-scroll chaotic systems.

As the master system, we consider the Sampath four-scroll chaotic system given by

\[
\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + bx_2x_3 \\
\dot{x}_2 &= -10x_2^3 - x_2 + 4x_1x_3 \\
\dot{x}_3 &= cx_3 - x_1x_2
\end{aligned}
\]  

(26)

where \(x_1, x_2, x_3\) are the state variables and \(a, b, c\) are positive parameters.

As the slave system, we consider the Sampath four-scroll chaotic system given by

\[
\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + by_2y_3 + u_1 \\
\dot{y}_2 &= -10y_2^3 - y_2 + 4y_1y_3 + u_2 \\
\dot{y}_3 &= cy_3 - y_1y_2 + u_3
\end{aligned}
\]  

(27)

where \(y_1, y_2, y_3\) are the state variables and \(u_1, u_2, u_3\) are the controls.

The hybrid chaos synchronization error is defined by

\[
\begin{aligned}
e_1 &= y_1 - x_1 \\
e_2 &= y_2 + x_2 \\
e_3 &= y_3 - x_3
\end{aligned}
\]  

(28)

Then the error dynamics is obtained as

\[
\begin{aligned}
\dot{e}_1 &= a(e_2 - e_1) - 2ax_2 + b(y_2y_3 - x_2x_3) + u_1 \\
\dot{e}_2 &= -e_2 - 10(y_2^3 + x_2^3) + 4(y_1y_3 + x_1x_3) + u_2 \\
\dot{e}_3 &= ce_3 - y_1y_2 + x_1x_2 + u_3
\end{aligned}
\]  

(29)

In matrix form, we can write the error dynamics (28) as

\[
\dot{e} = Ae + \phi(x, y) + u,
\]  

(30)

where

\[
A = \begin{bmatrix}
-a & a & 0 \\
0 & -1 & 0 \\
0 & 0 & c
\end{bmatrix}, \quad \phi(x, y) = \begin{bmatrix}
-2ax_2 + b(y_2y_3 - x_2x_3) \\
-10(y_2^3 + x_2^3) + 4(y_1y_3 + x_1x_3) \\
-cy_3 - y_1y_2 + x_1x_2
\end{bmatrix}, \quad u = \begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
\]  

(31)

First, we set \(u\) as

\[
u(t) = -\phi(x, y) + Bv(t),
\]  

(32)

where \(B\) is selected such that \((A, B)\) is completely controllable.

We choose \(B\) as
\[
B = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]  

(33)

We choose the parameters of the Sampath four-scroll systems as in the chaotic case, viz.

\[a = 3, \ b = 14, \ c = 3.9\]  

(34)

The sliding mode variable is selected as

\[s = Ce = \begin{bmatrix} 6 & 1 & -8 \end{bmatrix} e = 6e_1 + e_2 - 8e_3\]  

(35)

which renders the sliding motion globally asymptotically stable.

Next, we take the sliding mode gains as

\[k = 6 \quad \text{and} \quad q = 0.2.\]  

(36)

From Eq. (16) of Section 3, we obtain the novel sliding control \(v\) as

\[v = 18e_1 + 23e_2 - 79.2e_3 + 0.2s^2 \text{sgn}(s)\]  

(37)

As an application of Theorem 1 to the identical Zhu chaotic systems, we obtain the following main result of this section.

**Theorem 2.** The identical Sampath four-scroll chaotic systems (26) and (27) are globally and asymptotically hybrid synchronized for all initial conditions \(x(0), y(0) \in \mathbb{R}^3\) with the sliding controller \(u\) defined by (32), where \(\varphi(x, y)\) and \(B\) are defined by (31) and is defined by (37).

For numerical simulation, we take the parameter values as in the chaotic case, i.e.

\[a = 3, \ b = 14, \ c = 3.9\]

As an initial condition for the master system (25), we take

\[x_1(0) = 6.3, \ x_2(0) = 12.6, \ x_3(0) = -7.5\]

As an initial condition for the slave system (26), we take

\[y_1(0) = 18.9, \ y_2(0) = -11.2, \ y_3(0) = -15.3\]
Figures 5-7 depicts the hybrid chaos synchronization of the identical Sampath four-scroll chaotic systems. Figure 8 depicts the time-history of the hybrid synchronization errors.

6. CONCLUSIONS

In this paper, a novel sliding mode controller has been designed for the anti-synchronization of identical chaotic systems. Lyapunov stability theory has been used to prove this main result of the work. Next, as an application of the main result, a sliding controller has been designed for achieving hybrid chaos synchronization of identical Sampath four-scroll chaotic systems (2015). Numerical simulations using MATLAB have been provided to illustrate phase portraits of the Sampath four-scroll system and the novel sliding mode controller for the hybrid chaos synchronization of identical Sampath four-scroll chaotic systems.

References


