On Generalized Second $\Phi$-Variation of Normed Space Valued Maps

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Abstract: In this article we present a Young-type generalization of the concept of second variation for normed space valued functions defined on an interval $[a, b] \subset \mathbb{R}$. We show that a function $f$ which take values on a Banach space $X$, is of bounded second $\Phi$-variation, in the sense of Young, if and only if it can be expressed as the (Bochner) integral of a function of bounded (first) $\Phi$-variation. This extend results of \cite{11} and \cite{13}.

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1. INTRODUCTION

In 1881 C. Jordan \cite{7} introduced the class of real valued functions of bounded variation and established the relation between these functions and the monotonic ones; namely, a function of bounded variation can be expressed as the difference of two increasing functions. Thus, the Dirichlet Criterium for the convergence of the Fourier series of a monotone function applies to the class of functions of bounded variation. This, in turn, has motivated the study of solutions of nonlinear equations that describe concrete physical phenomena in which, often, functions of bounded variation intervene.

The interest generated by this notion has lead to some generalizations of the concept, mainly, intended to the search of a bigger class of functions whose elements have pointwise convergent Fourier series. As in the classical case, these generalizations have found many applications in the study of certain differential and integral equations.

In 1908, De La Vallee Poussin \cite{5} obtained a generalization of the concept of function of bounded variation introducing the notion of function of second bounded variation on a closed interval. He showed that if $f$ is of second bounded variation on $[a, b]$ then it is absolutely continuous on $[a, b]$ and it can be expressed as a difference of two convex functions. A few years later, in 1911, \cite{10}, F. Riesz proved that a function $f$ is of bounded second variation on $[a, b]$ if, and only if, it is the definite Lebesgue integral of a function of bounded variation $F$. More recently, in 1983, A. M. Russell and C. J. F. Upton (\cite{11}) obtained a similar result for functions of bounded second $p$th-variation $(1 \leq p \leq \infty)$, in the sense of Wiener. On the other hand, in 1937, L. C. Young (\cite{13}) gave a generalization of the concept of function of bounded variation by introducing the notion of $\Phi$-variation of a function, this concept, in turn, was generalized by V. Chistiakov, \cite{3} for functions which take values in a linear normed space. In this article we introduce the notion of function of bounded second $\Phi$-variation in the sense of Young. We show that a function $f$ is of bounded second $\Phi$-variation in the sense of Young if and only if it is the integral of a function of bounded $\Phi$-variation.

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In this paper we will use the following standard notation (see [4]): \( \mathcal{N} \) will denote the set all continuous convex functions \( \Phi : [0, +\infty) \to [0, +\infty) \) such that \( \Phi(0) = 0 \) if and only if \( \rho = 0 \), and \( \mathcal{N}_* \) the set of all functions \( \Phi \in \mathcal{N} \), for which the Orlicz condition (also called \( \infty \) condition) holds:

\[
\lim_{\rho \to +\infty} \frac{\Phi(\rho)}{\rho} = +\infty.
\] (1.1)

2. PRELIMINARIES

In the sequel \( X \) will denote a normed space with norm \( \| \cdot \| \).

**Definition 2.1:** ([3]). The (generalized) \( \Phi \)-variation of a map \( f : [a, b] \to X \) is

\[
V_{\Phi}(f, [a,b]) = V_{\Phi}(f, [a,b], X) := \sup_{\xi} \sum_{i=1}^{n} \Phi(\|f(t_i) - f(t_{i-1})\|)
\] (2.1)

where the supremum is taking over the set of all partitions \( \xi \) of the interval \([a, b]\).

It is known that the functional \( V_{\Phi} \) is nondecreasing, that is, \( V_{\Phi}(f, [a, b]) \leq V_{\Phi}(f, [c, d]) \) if \([a, b] \subseteq [c, d]\); it is semiadditive: \( V_{\Phi}(f, [a, c]) + V_{\Phi}(f, [c, b]) \leq V_{\Phi}(f, [a, b]) \) for all \( a < c < b \); and it is sequentially lower semicontinuous. The set of all functions \( f \in X^{\infty} \) for which \( V_{\Phi}(f, [a, b]) < +\infty \) is not necessarily a linear space, but it is a convex subset of \( X^{\infty} \) and \( V_{\Phi}(\cdot, [a, b]) \) is a convex functional on it.

The class \( W_{\Phi}(f, [a, b], X) := \{ f \in X^{\infty} : \exists \lambda > 0, V_{\Phi}(f, \lambda) < \infty \} \) is a linear space, called the class of functions of bounded \( \Phi \)-variation, in the sense of Young. It can be equipped with the norm: \( \|f\| := f(a) + \rho_{\Phi}(f) \), where

\[
\rho_{\Phi}(f) = \inf \{ \lambda > 0 : V_{\Phi}(f, \lambda) < 1 \}.
\]

More on \( W_{\Phi}(f, [a, b], X) \) can be seen in [3].

By \( \pi_{a} [a, b] \) we will denote the class of all partitions of an interval \([a, b]\) that contain at least two points \( t, s \in (a, b) \).

**Definition 2.2:** Let \( f : [a, b] \to X \), let \( \xi = \{x_j\}_{j=0}^{m} \in \pi_{a} [a, b] \), and let \( \Phi \in \mathcal{N} \). We shall use the following notation:

\[
U(f; x_j, x_i) := \frac{f(x_j) - f(x_i)}{x_j - x_i}
\]

and define the second variation of a function \( f \) on \([a, b]\), in the sense of Young, as

\[
V_{\Phi}^2(f; [a, b], X) = \sup_{\xi \in \pi_{a} [a, b]} \tilde{g}_{\Phi}^2(f; [a, b], \xi) \]

where

\[
\tilde{g}_{\Phi}^2(f; [a, b], \xi) := \sum_{j=1}^{m-2} \Phi(\|U(f; x_j + 2, x_{j+1}) - U(f; x_j, x_{j+1})\|).
\]

If \( V_{\Phi}^2(f; [a, b], X) < \infty \) we will say that the function \( f \) of bounded second \( \Phi \)-variation, in the sense of Young, and we will write \( f \in BV_{\Phi}^2 ([a, b], X) \).

**Remark 2.3:** If \( \Phi \) does not satisfy condition (1.1); that is, if \( \lim_{\rho \to +\infty} \Phi(\rho)/\rho \) exists, then it is readily seen that the space \( BV_{\Phi}^2 ([a,b], \mathbb{R}) \) is equal to the classical space of functions of second bounded variation. Thus, from now on we will assume that all the functions \( \Phi \) considered are in \( \mathcal{N}_* \).
Example 2.4: Let \( f: [a, b] \to C_0 \) in \([a, b]\) defined by \( f(t) = \left( \frac{|t|}{n} \right)_{n \in \mathbb{N}} \).

If \( 0 < a < b \), for every partition \( \xi = \{x_i\}_{i=0}^{m} \in \pi_{4}[a, b] : \)

\[
\sum_{j=1}^{m-2} \Phi \left( \left\| \frac{f(x_{j+2}) - f(x_{j+1})}{x_{j+2} - x_{j+1}} - \frac{f(x_{j}) - f(x_{j-1})}{x_{j} - x_{j-1}} \right\| \right)
\]

\[
= \sum_{j=1}^{m-2} \Phi \left( \left\| \frac{(x_{j+2} - x_{j+1}) - (x_{j+1} - x_{j})}{x_{j+2} - x_{j+1}} \right\| \right)
\]

\[
= \sum_{j=1}^{m-2} \Phi \left( \left\| \frac{(x_{j+2} - x_{j+1})}{x_{j+2} - x_{j+1}} \right\| \right)
\]

which means \( f \in BV^2_\phi([a, b], X) \) and \( V^2_\phi(f; [a, b], X) = 0. \)

A similar estimation holds if \( a < b < 0. \)

On the other hand, if \( a > 0 \) and \( \xi = \left\{ -a, \frac{a}{2}, a \right\} \), then

\[
\Phi \left( \left\| \frac{f(a) - f(a/2)}{a - a/2} - \frac{f(0) - f(-a)}{0 - (-a)} \right\| \right)
\]

\[
= \Phi \left( \left\| \frac{(a/2) - (a/2)}{a/2} - \frac{0 - (a/2)}{0 - (-a)} \right\| \right)
\]

\[
= \Phi \left( \left\| \frac{1}{n} - \frac{1}{n} \right\| \right)
\]

Therefore \( 0 < \Phi(2) \leq V^2_\phi(f; [-a, a]). \)

Lemma 2.5: If \( f, g \in BV^2_\phi([a, b], X) \), \( \lambda \) is a complex constant with \( |\lambda| \leq 1 \) and \( \alpha, \beta \) are real numbers such that \( \alpha + \beta = 1 \), then

(i) \( V^2_\phi(\lambda f; [a, b], X) \leq \lambda V^2_\phi(f; [a, b], X), \)

(ii) \( V^2_\phi \) is convex in the function argument; that is

\[
V^2_\phi(\alpha f + \beta g; [a, b], X) \leq \alpha V^2_\phi(f; [a, b], X) + \beta V^2_\phi(g; [a, b], X)
\]

Proof: The proof is a consequence of the convexity of both the function \( \Phi \) and the norm of the space \( X. \)

Definition 2.6: ([4]). A mapping \( f: [a, b] \to X \) is called absolutely continuous if there exists a function \( \delta: (0, 1) \to (0, 1) \) such that for any \( \epsilon > 0 \), any \( n \in \mathbb{N} \) and any finite collection of points \( \{a_i, b_i\}_{i=1}^{n} \subset [a, b] \)

such that \( a_1 < b_1 \leq a_2 < b_2 \ldots \leq a_n < b_n \), the condition \( \sum_{i=1}^{n} (b_i - a_i) < \delta(\epsilon) \) implies \( \sum_{i=1}^{n} \| f(b_i) - f(a_i) \| < \epsilon. \)
Lemma 2.7: Let $f \in BV^2_{\phi}([a, b], X)$. Then

(i) $U(f; y_0, y_1)$ is bounded for all $y_0, y_1 \in [a, b]$,

(ii) $f$ is absolutely continuous on $[a, b]$,

(iii) If $x_0, y_0 \in [a, b]$ with $x_0 \neq y_0$ then $U(f; x_0, x)$ is continuous at $x = y_0$.

Proof: (i) The proof is similar to the respective one given in [11]. Let $y_0, y_1 \in [a, b]$, and let $c \in (a, b)$. The proof depends on the location of $y_0, y_1$ with respect to $a, b$ and $c$.

**Case 1:** $a < y_0 < c < y_1 < b$.

In this case, for $y_2 \in (y_1, b)$, we have

$$
\|U(f; y_0, y_1)\| \leq \|U(f; y_0, y_1) - U(f; y_2, b)\| + \|U(f; y_2, b) - U(f; a, c)\| + \|U(f; a, c)\|
$$

$$
= \Phi^{-1}(\Phi(\|U(f; y_0, y_1) - U(f; y_2, b)\|)) + \Phi^{-1}(\|U(f; y_2, b) - U(f; a, c)\|) + \|U(f; a, c)\|
$$

$$
\leq 2\Phi^{-1}(V^2_{\phi}(f; [a, b], X)) + \|U(f; a, c)\|.
$$

**Case 2:** $a < y_0 < c < y_1 = b$.

Here, for $y_2 \in (a, y_0)$ :

$$
\|U(f; y_0, y_1)\| \leq \|U(f; y_0, y_1) - U(f; a, y_2)\| + \|U(f; a, y_2) - U(f; c, b)\| + \|U(f; c, b)\|
$$

$$
= \Phi^{-1}(\Phi(\|U(f; y_0, y_1) - U(f; a, y_2)\|)) + \Phi^{-1}(\|U(f; a, y_2) - U(f; c, b)\|) + \|U(f; c, b)\|
$$

$$
\leq \Phi^{-1}(V^2_{\phi}(f; [a, b], X)) + \Phi^{-1}(V^2_{\phi}(f; [a, b], X)) + \|U(f; c, b)\|
$$

$$
= 2\Phi^{-1}(V^2_{\phi}(f; [a, b], X)) + \|U(f; c, b)\|.
$$

In any other case one proceeds in a similar fashion.

Since $c$ is arbitrary but fixed $U(f; y_0, y_1)$ must be bounded.

(ii) By (i) there exists $M \geq 0$ such that $\|U(f; y_0, y_1)\| \leq M$ for all $y_0, y_1 \in [a, b]$.

Therefore $f$ is Lipschitz, and hence, absolutely continuous.

(iii) It is immediate.

Lemma 2.8: Let $f \in BV^2_{\phi}([a, b], X)$ and $c \in (a, b)$, then $f \in BV^2_{\phi}([a, c]) \cap BV^2_{\phi}([c, b])$ and

$$
V^2_{\phi}(f; [a, c], X) + V^2_{\phi}(f; [c, b], X) \leq V^2_{\phi}(f; [a, b], X).
$$

(2.2)

The proof is elementary and will be omitted.

Remark 2.9: Inequality (2.2) is the best possible; that is, it cannot be replaced by an equality as can be readily verified by considering the example 2.4, with $a = -1, b = 1, c = 0$ and taking $\xi = \{-1, 0, 1/2, 1\}$.

3. MAIN RESULTS

Theorem 3.1: Suppose $f \in BV^2_{\phi}([a, b], X)$ and let $F(x) := \int_a^x f(t) \, dt$ (Bochner integral on the right hand side).

Then $F \in BV^2_{\phi}([a, b])$ and

$$
V^2_{\phi}(F; [a, b], X) \leq V^2_{\phi}(f; [a, b], X).
$$
Proof: For a given partition \( \xi \in \pi_{i} \ [a, b] \), we have

\[
\dot{\Omega}_{2}^{2}(F;[a,b],\xi) = \sum_{j=1}^{m-2} \Phi \left( \left\| \frac{F(x_{j+1}) - F(x_{j})}{x_{j+1} - x_{j}} - \frac{F(x_{j}) - F(x_{j-1})}{x_{j} - x_{j-1}} \right\| \right)
\]

\[
= \sum_{j=1}^{m-2} \Phi \left( \left\| \int_{x_{j}}^{x_{j+1}} f(t) \frac{dt}{x_{j+1} - x_{j}} - \int_{x_{j-1}}^{x_{j}} f(t) \frac{dt}{x_{j} - x_{j-1}} \right\| \right).
\]

Thus, putting \( u = \frac{t - x_{j-1}}{x_{j+1} - x_{j}} \), \( v = \frac{t}{x_{j}} \), and making use of Jensen inequality, we obtain

\[
\dot{\Omega}_{2}^{2}(F;[a,b],\xi) = \sum_{j=1}^{m-2} \Phi \left( \left\| \int_{0}^{1} f(u[x_{j+1} - x_{j-1}] + x_{j+1}) du - \int_{0}^{1} f(v[x_{j} - x_{j-1}] + x_{j}) dv \right\| \right)
\]

\[
= \sum_{j=1}^{m-2} \Phi \left( \left\| \int_{0}^{1} f(\tau[x_{j+1} - x_{j-1}] + x_{j+1}) - f(\tau[x_{j} - x_{j-1}] + x_{j}) d\tau \right\| \right)
\]

\[
\leq \sum_{j=1}^{m-2} \int_{0}^{1} \Phi \left( \left\| f(\tau[x_{j+1} - x_{j-1}] + x_{j+1}) - f(\tau[x_{j} - x_{j-1}] + x_{j}) \right\| d\tau \right)
\]

\[
= \int_{0}^{1} \sum_{j=1}^{m-2} \Phi \left( \left\| f(\tau[x_{j+1} - x_{j-1}] + x_{j+1}) - f(\tau[x_{j} - x_{j-1}] + x_{j}) \right\| d\tau \right)
\]

\[
= \int_{0}^{1} V_{\Phi}(f;[a,b]) du = V_{\Phi}(f;[a,b]).
\]

Since this inequality holds for any partition \( \xi \in \pi_{i} \ [a, b] \), we conclude that

\[
V_{\Phi}^{2}(F;[a,b],X) \leq V_{\Phi}(f; [a, b]).
\]

Lemma 3.2: Let \( E \) be a dense subset of \([a, b]\), let \( g \) be a function defined on \( E \) and let \( K \) be a positive constant such that

\[
\sum_{j=0}^{m-1} \Phi(\left\| g(x_{j+1}) - g(x_{j}) \right\|) \leq K,
\]

where \( a \leq x_{0} < x_{1} < \cdots < x_{m-1} < x_{m} \leq b \) and \( x_{j} \in E \) for \( j = 0, 1, 2, \cdots, m \). Then, \( g_{E}(x - 0) \) exists for all \( x \in (a, b) \setminus E \), where

\[
g_{E}(x - 0) = \lim_{h \to 0, x \in E} g(x - h).
\]

Proof: Suppose that the result is false; that is, suppose that there is a \( t \in (a, b) \setminus E \) such that

\[
\lim_{h \to 0, t \in E} g(t - h) = \lim_{s \to 0, s \in E} g(s) \text{ does not exist.}
\]

Thus, if we make

\[
\Lambda := \limsup_{s \to 0, s \in E} g(s) \text{ and } \gamma := \liminf_{s \to 0, s \in E} g(s)
\]
then \( \gamma < \Lambda \). Therefore, we can find sequences \( (x_n)_{n \in \mathbb{N}} \) and \( (x_n^\prime)_{n \in \mathbb{N}} \) in \( E \) such that, for all \( n: x_n^\prime < x_n^\prime < x_n < x_n^\prime < t \),

\[
\lim_{x \to a^+} x_n^\prime = \lim_{x \to a^-} x_n^\prime = t,
\]

\[
\lim_{x \to a^+} g(x_n^\prime) = \Lambda, \quad \text{and} \quad \lim_{x \to a^-} g(x_n^\prime) = \lambda.
\]

If \( \Lambda \) and \( \lambda \) are finite, then, for \( \epsilon = \frac{\Lambda - \lambda}{3} \), we can choose \( N \in \mathbb{N} \)

\[
\| g(x_n^\prime) - g(x_n^\prime) \| > \epsilon \quad \text{for all } n > N.
\] (3.2)

If \( \Lambda \) or \( \lambda \) are not finite, (3.2) holds for any \( \epsilon \). It follows from (3.2) that

\[
\sum_{n=N+1}^{N+1} \Phi(\| g(x_n^\prime) - g(x_n^\prime) \|) > \sum_{n=N+1}^{N+1} \Phi(\epsilon) = k \Phi(\epsilon)
\]

for all \( k > 0 \), which contradicts (3.1).

**Lemma 3.3:** Let \( E \) be a dense subset of \([a, b]\), let \( g \) be a function defined on \( E \) and let \( K \) be a positive constant such that

\[
\| g(x_j^\prime) - g(x_j^\prime) \| \leq K,
\]

where \( a \leq x_0 < x_1 < \ldots < x_{m-1} < x_m \leq b \) and \( x_j \in E \) for \( j = 0, 1, 2, \ldots, m \). Then, \( g_E(x + 0) \) exists for all \( x \in (a, b] \setminus E \), where

\[
g_E(x + 0) = \lim_{h \to 0, x + 0 \in E} g(x + h).
\]

**Proof:** The proof is similar to the one given for lemma 3.2.

**Theorem 3.4:** Let \( F \in BV_\Phi([a, b], X) \). Then there is a function \( f \in BV_\Phi([a, b]) \) such that \( F' = f \), Lebesgue-a.e. and

\[
V_\Phi(f; [a, b], X) = V_\Phi(F; [a, b], X).
\] (3.4)

**Proof:** Lemma 2.7 guarantee us that \( F \) is absolutely continuous. Hence, it is strongly differentiable a.e. (see [1]), with derivative strongly measurable. Let \( E \) be a set of zero Lebesgue measure such that \( F' \) exists at every point of \( \mathbb{R}^+ = [a, b] \setminus E \).

Consider points \( (x_j^\prime)_0^n \in [a, b] \cap \mathbb{R}^+ \) such that \( a \leq x_0 < x_1 < \ldots < x_m \leq b \) and let \( h_0, h_1, \ldots, h_m \) be positive real numbers such that \( \frac{x_{j+1} - x_j}{2} \geq h_j \) for all \( j = 0, \ldots, m - 1 \) and \( x_{m-1} + h_{m-1} < x_m - h_m \). Next define a partition \( \{y_j\}_{j=0}^{2m} \) of \([a, b]\) as

\[
y_j = \begin{cases} 
  x_{j+1/2} & \text{IF } 0 \leq j \leq m - 1 \text{ is even} \\
  x_{j+1} + h_{j+1} & \text{IF } 0 \leq j \leq m - 1 \text{ is odd} \\
  x_m - h_m & \text{IF } j = 2m \\
  x_m & \text{IF } j = 2m + 1
\end{cases}
\]
Then
\[
\sum_{j=0}^{m-2} \Phi \left( \left\| \frac{F(x_{j+1} + h_{j+1}) - F(x_{j+1})}{h_{j+1}} - \frac{F(x_j + h_j) - F(x_j)}{h_j} \right\| \right)
+ \Phi \left( \left\| \frac{F(x_m) - F(x_m - h_m)}{h_m} - \frac{F(x_{m-1} + h_{m-1}) - F(x_{m-1})}{h_{m-1}} \right\| \right)
= \sum_{j=0}^{m-2} \Phi \left( \left\| \frac{F(y_{2(j+1)+1}) - F(y_{2(j+1)})}{y_{2(j+1)+1} - y_{2(j+1)+1}} - \frac{F(y_{2(j+1)+1} - y_{2(j+1)+2})}{y_{2(j+1)+1} - y_{2(j+1)+2}} \right\| \right)
+ \Phi \left( \left\| \frac{F(y_{2m+1}) - F(y_{2m})}{y_{2m+1} - y_{2m}} - \frac{F(y_{2m+1} - y_{2m-1})}{y_{2m+1} - y_{2m-1}} \right\| \right)
= \sum_{k=0}^{m-1} \Phi \left( \left\| \frac{F(y_{k+2}) - F(y_{k+1})}{y_{k+1} - y_{k+2}} - \frac{F(y_{k}) - F(y_{k-1})}{y_{k} - y_{k-1}} \right\| \right)
\leq V^2_\Phi(F;[a,b],X).
\]

Making \( h_j \to 0 \), one gets
\[
\sum_{j=1}^{m-1} \Phi \left( \left\| F'(x_{j+1}) - F'(x_j) \right\| \right) = V^2_\Phi(F;[a,b],X),
\] (3.5)

therefore, by lemmas 3.2 and 3.3, \( F'_E(x - 0) \) and \( F'_E(x + 0) \) exists for all \( x \in (a, b) \) \(\not\in\) \( \mathbb{R}^\prime \).

Define now the function
\[
f(x) : = \begin{cases} 
F'(x) & \text{cando } x \in \mathbb{R}' , \\
F'_E(x - 0) & \text{cando } x \in (a, b) \setminus \mathbb{R}' , \\
F'_E(a) & \text{si } x = a \text{ y } a \not\in \mathbb{R}' ,
\end{cases}
\] (3.6)

Clearly, \( F' = f \) almost everywhere. We need just to verify that \( f \) satisfies (3.4). Let \( \xi : = \{ x_j \}_{j=0}^{m} \) be a partition of \( [a, b] \). Suppose that there is exactly a point \( a \neq x_k \in \xi \) such that \( x_k \not\in \mathbb{R}' \). In this case, we can choose \( x_k' \in \mathbb{R}' \) such that \( x_{k-1} < x_k' < x_k \). Put \( \xi' : = \{ x_0', x_1', ..., x_{k-1}', x_k' \} \). Observe that
\[
\lim_{x_k \to x_k'} f(x_k) = \lim_{x_k \to x_k'} f(x_k') = F'_E(x_k - 0)
\]

Moreover, for each \( x \in (a, b) \setminus \mathbb{R}' \),
\[
f(x) = F'_E(x - 0) = \lim_{h \to 0^+} F'(x - h) = \lim_{h \to 0^+} F(x - h) = f_E(x - 0).
\] (3.7)

Therefore
\[
\sum_{j=0}^{k-2} \Phi \left( \left\| f(x_{j+1}) - f(x_j) \right\| \right) + \Phi \left( \left\| f(x_k') - f(x_{k-1}) \right\| \right) + \Phi \left( \left\| f(x_{k+1}) - f(x_k') \right\| \right)
+ \sum_{j=k+1}^{m-1} \Phi \left( \left\| f(x_{j+1}) - f(x_j) \right\| \right) \leq V^2_\Phi(F;[a,b],X).
\]
and taking the limit is \( x'_{k} \to x_{k} \) we have

\[
\sum_{j=0}^{k-2} \Phi(\| f(x_{j+1}) - f(x_{j}) \|) + \Phi(\| f(x_{k}) - f(x_{k-1}) \|)
\]

\[+ \Phi(\| f(x_{k+1}) - f(x_{k}) \|) + \sum_{j=k+1}^{m-1} \Phi(\| f(x_{j+1}) - f(x_{j}) \|) \leq V_{\Phi}^{2}(F;[a,b],X).\]

If, on the other hand, \( x_{0} = a \) is the only point of \( \xi \) not in \( \mathbb{B}' \), then we may consider a collection \( \{ x'_{0}, x_{1}, \ldots, x'_{k-1}, x'_{k}, x_{k+1}, \ldots, x_{m} \} \), where \( x'_{0} \in \mathbb{B}' \), \( a < x'_{0} < x_{1} \). Then

\[
\lim_{x_{0} \to x'_{0}} f(x'_{0}) = f_{E}(x_{0} + 0),
\]

and

\[
f(a) = F_{E}(a + 0) = \lim_{h \to 0^{+}} F'(a + h) = \lim_{h \to 0^{+}} f(a + h) = f_{E}(a + 0). \tag{3.8}
\]

Thus, proceeding as above, we get

\[
\Phi(\| f(x_{1}) - f(x'_{0}) \|) + \sum_{j=1}^{m-1} \Phi(\| f(x_{j+1}) - f(x_{j}) \|) \leq V_{\Phi}^{2}(F;[a,b],X),
\]

and taking limit when \( x'_{0} \to x_{0} \):

\[
\Phi(\| f(x_{1}) - f(x_{0}) \|) + \sum_{j=1}^{m-1} \Phi(\| f(x_{j+1}) - f(x_{j}) \|) \leq V_{\Phi}^{2}(F;[a,b],X),
\]

To complete the proof that \( f \) is of bounded \( \Phi \)-variation and satisfies (3.4), let \( \xi : = \{ x_{j} \}_{j=0}^{m} \in \pi_{\Phi} \left( [a, b] \right) \) arbitrary, and set \( \xi' = \{ x'_{0}, x_{1}, \ldots, x'_{k-1}, x'_{k}, x'_{k+1}, \ldots, x'_{m} \} \) where: \( x'_{j} = x_{j} \) if \( x_{j} \notin \mathbb{B}' \); if \( a \neq x_{j} \notin \mathbb{B}' \). Then, pick \( x_{j-1} < x'_{j} \) \( \leq x'_{j} \), whereas if \( a \notin \mathbb{B}' \) take \( a < x'_{0} < x_{1} \).

From (3.8) we obtain

\[
\sum_{j=0}^{m-1} \Phi(\| f(x_{j+1}) - f(x_{j}) \|) \leq V_{\Phi}^{2}(F;[a,b],X).
\]

Hence, by taking the supremum over all partitions \( \xi \) of \([a, b] \)

\[
V_{\Phi}(f; [a, b]) \leq V_{\Phi}^{2}(F; [a, b], X). \tag{3.9}
\]

To prove the reverse inequality we use the fact that \( F \) is absolutely continuous and consequently it has strong derivative almost everywhere (Lebesgue) on \([a, b] \).

Thus

\[
F(x) = F(a) + \int_{a}^{x} F'(t)dt = F(a) + \int_{a}^{x} f(t)dt.
\]

Therefore, by theorem (3.1), the function \( \Phi(x) : = F(x) - F(a) \) is also of bounded second \( \Phi \)-variation, in the sense of Young, and

\[
V_{\Phi}^{2}(F; [a, b], X) \leq V_{\Phi}(f; [a, b]).
\]

This last inequality and (3.9) imply (3.4).
REFERENCES


